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Consistent section-averaged equations of quasi-one-dimensional laminar flow

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Section-averaged equations of motion, widely adopted for slowly varying flows in pipes, channels and thin films, are usually derived from the momentum integral on a heuristic basis, although this formulation is affected by known inconsistencies. We show that starting from the energy rather than the momentum equation makes it become consistent to first order in the slowness parameter, giving the same results that have been provided until today only by a much more laborious two-dimensional solution. The kinetic-energy equation correctly provides the pressure gradient because with a suitable normalization the first-order correction to the dissipation function is identically zero. The momentum equation then correctly provides the wall shear stress. As an example, the classical stability result for a free falling liquid film is recovered straightforwardly.

Key words: general fluid mechanics, instability, shallow water flows

1. Momentum versus kinetic-energy section-averaged equations

Incompressible flow in a duct of slowly varying cross-section is classically described by section-averaged equations of mass, momentum and kinetic-energy balance, only two of the three being independent. In most high-Reynolds-number applications, the velocity profile in the cross-section is treated as being flat; this plug-flow assumption is appropriate for an inviscid fluid, but becomes inconsistent when dissipation and head loss are involved, because a viscous fluid must have zero velocity at the wall. From introductory textbooks we learn that a better approximation, especially when the flow is laminar, can be gained through correction coefficients based on the Poiseuille velocity profile in a straight duct. We are not generally warned, however, to the effect of two different sets of correction coefficients being generated for the momentum and kinetic-energy (Bernoulli) equations: such equations can no longer be derived from one another. Once we begin to know better, the question arises as to which of the two sets of coefficients (if either) is in any reasonable sense correct.

Let us consider the example of a two-dimensional laminar flow with longitudinal velocity $u(x, y, t)$ in a symmetric plane duct, the symmetry axis ($u_y = 0$) being at $y = 0$ and the wall ($u = 0$) at $y = h(x, t)$. The integral mass, momentum and kinetic-energy

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balance equations for this flow, in dimensionless form, are

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}\int_0^h u\,\mathrm{d}y = 0, \tag{1.1a}$$

$$\frac{\partial}{\partial t}\int_0^h u\,\mathrm{d}y + \frac{\partial}{\partial x}\int_0^h u^2\,\mathrm{d}y + \int_0^h P_x\,\mathrm{d}y = -\tau = \frac{1}{Re}u_y(h), \tag{1.1b}$$

$$\frac{\partial}{\partial t}\int_0^h \frac{u^2}{2}\,\mathrm{d}y + \frac{\partial}{\partial x}\int_0^h \frac{u^3}{2}\,\mathrm{d}y + \int_0^h uP_x\,\mathrm{d}y = -f = -\frac{1}{Re}\int_0^h u_y^2\,\mathrm{d}y, \tag{1.1c}$$

where τ is the wall shear stress and f the dissipation. P is a modified pressure defined as the sum of the thermodynamic pressure and gravitational potential, introduced as customary in an incompressible flow to separate dynamic effects from those of gravity. Omission of the square of transverse velocity from (1.1c) is allowed because this equation will be expanded to no more than first order in a small slope parameter in the following.

If P_x is now approximated by a constant over the cross-section and u by its straight-duct Poiseuille expression $u=(3U/2)(1-y^2/h^2)$, with $U(x,t)$ denoting the sectional mean velocity, all the integrals can be calculated easily:

$$\frac{\partial h}{\partial t} + \frac{\partial(hU)}{\partial x} = 0, \tag{1.2}$$

$$\frac{\partial(hU)}{\partial t} + \frac{6}{5}\frac{\partial(hU^2)}{\partial x} + hP_x = -\frac{3U}{Reh}, \tag{1.3}$$

$$\frac{6}{5}\frac{\partial}{\partial t}\frac{hU^2}{2} + \frac{54}{35}\frac{\partial}{\partial x}\frac{hU^3}{2} + hUP_x = -\frac{3U^2}{Reh}. \tag{1.4}$$

However, multiplying both sides of (1.3) by U and combining the result with (1.2) so as to generate the corresponding kinetic-energy equation give

$$\frac{\partial}{\partial t}\frac{hU^2}{2} + \frac{\partial}{\partial x}\frac{hU^3}{2} + \frac{U}{5}\frac{\partial(hU^2)}{\partial x} + hUP_x = -\frac{3U^2}{Reh}.$$

Clearly, (1.3) and (1.4) cannot both be right.

2. Consistent formulation

It does not take much to realize that the flaw in the above derivation was the undiscriminating substitution of the Poiseuille profile everywhere. Indeed in a particular application to which we shall return below (free-surface and falling-film instabilities), it has long been known in the literature that (1.3) gives slightly wrong growth rates.

The consistent formulation of this problem is a multiple-scale expansion, in which $\partial h/\partial t$ and $\partial h/\partial x$, though not h, are assumed to be proportional to an arbitrarily small parameter ε. In other words, t and x are replaced by T/ε and X/ε, with T and X being $O(1)$ rescaled coordinates. The variable u is then represented as a power series $u=u^{(0)}+\varepsilon u^{(1)}+\cdots$, with $u^{(0)}=(3U/2)(1-y^2/h^2)$, and a straightforward (but lengthy) expansion of the full two-dimensional problem can be continued to any desired order. (The term $u^{(1)}$ eventually turns out to be proportional to the Reynolds number; thus, in a laminar large-Reynolds-number flow the product εRe has to be small compared to 1.)

In order to extract as much information as possible from a multiple-scale expansion applied to the integral equations (1.1), we can observe the following. (a) The otherwise arbitrary coefficient $U(X,T)$ can always be chosen to be equal to the true mean velocity, so that the flow rate associated with $u^{(1)}$ is identically zero. This takes care of the first-order correction to (1.1*a*). (*b*) The first two terms of (1.1*b*) and (1.1*c*) are already $O(\varepsilon)$, and only $u^{(0)}$ needs appear in their body. (*c*) All terms in the transverse-momentum equation (not written here) are $O(\varepsilon^2)$ or higher; $P_x^{(1)}$ is then constant over the cross-section just like $P_x^{(0)}$. The ensuing first-order equations are

$$\frac{\partial}{\partial T}\int_0^h u^{(0)}\,\mathrm{d}y + \frac{\partial}{\partial X}\int_0^h u^{(0)2}\,\mathrm{d}y + hP_x^{(1)} = -\tau^{(1)} = \frac{1}{Re}u_y^{(1)}(h), \tag{2.1}$$

$$\frac{\partial}{\partial T}\int_0^h \frac{u^{(0)2}}{2}\,\mathrm{d}y + \frac{\partial}{\partial X}\int_0^h \frac{u^{(0)3}}{2}\,\mathrm{d}y + hUP_x^{(1)} = -f^{(1)} = -\frac{2}{Re}\int_0^h u_y^{(0)}u_y^{(1)}\,\mathrm{d}y. \tag{2.2}$$

The puzzle is thus solved: (1.3) and (1.4) were lacking first-order corrections to shear stress and dissipation, respectively, which explains their incompatibility. It would seem as if the only way out of this difficulty were to patiently calculate the first-order velocity $u^{(1)}$, and indeed there are examples of this procedure in the literature (Ruyer-Quil & Manneville 1998), giving perfectly satisfactory results. What we want to point out in this paper is a fortunate simplification which was suggested to us by the minimum-dissipation property (Batchelor 1967, p. 227) of creeping flow, to which a straight-duct flow reduces for any Reynolds number: for any first-order correction $u^{(1)}$ (of zero flow rate) to the velocity profile, the first-order correction $f^{(1)}$ to the dissipation function is identically zero. In order to prove that this property is preserved by the present approximation the expression of $f^{(1)}$ in (2.2) can be manipulated as follows:

$$\frac{Re}{2}f^{(1)} = \int_0^h u_y^{(0)}u_y^{(1)}\,\mathrm{d}y = \left[u_y^{(0)}u^{(1)}\right]_0^h - \int_0^h u_{yy}^{(0)}u^{(1)}\,\mathrm{d}y$$

$$= u_y^{(0)}(h)u^{(1)}(h) - Re\,P_x^{(0)}\int_0^h u^{(1)}\,\mathrm{d}y = 0. \tag{2.3}$$

The first term is zero provided the boundary is either a solid ($u^{(1)}(h)=0$) or a stress-free ($u_y^{(0)}(h)=0$) surface; the last term is zero because so is the integral of $u^{(1)}$.

Knowing that the first-order dissipation $f^{(1)}$ is zero obviates the need to calculate the first-order velocity explicitly. Adding (2.2) multiplied by ε to the zeroth-order equation $hUP_x^{(0)} = -f^{(0)} = -3U^2/Reh$ gives back (1.4), with $P = P^{(0)} + \varepsilon P^{(1)}$. It follows that after all (1.4) was correct to first order in ε (while (1.3) was not), and can rightfully be used to calculate the (0+1)th-order pressure gradient, without needing to know the details of the first-order velocity profile.

3. An application example: the instability of a falling film

As a test, the one-dimensional energy equation can be verified to reproduce the result of Benjamin's (1957) and Yih's (1963) two-dimensional solutions for the long-wave linear instability of a free-surface liquid film falling down an inclined plane. Flow in this configuration is identical to the previous example of a variable duct, but with reversed boundary conditions ($u=0$ at $y=0$ and $u_y=0$ at $y=h$). Equations (1.1)–(1.4) apply unchanged; constancy of the thermodynamic pressure at the free surface leads the modified-pressure gradient P_x to be expressed in dimensional form as $P_x = \rho g(h_x\cos\theta - \sin\theta)$, where θ is the inclined plane's slope angle, or in dimensionless form as $P_x = Fr^{-2}(h_x - \tan\theta)$, with $Fr = U_{ref}/(gh_{ref}\cos\theta)^{1/2}$ being the Froude number.

If the equilibrium height h and mean velocity U are now assumed to be unity (i.e. their dimensional values have been used as h_{ref} and U_{ref}), the unperturbed uniform flow obeys $Fr^2 = Re \tan\theta/3$.

Despite it having long been known in the literature to give slightly wrong growth rates, the Shkadov (1967) equation based on (1.3) is widely adopted as a one-dimensional approximation to this problem on a purely empirical basis (Chang & Demehkin 2002). Benjamin's (1957) and Yih's (1963) small-wavenumber expansions, on the other hand, represented u as a regular power series in wavenumber k (more precisely, in the product kRe) and consequently expanded the complete two-dimensional Orr–Sommerfeld equation. The corresponding nonlinear long-wave approximation was introduced by Benney (1966). However, the simplicity of the section-averaged formulation was lost.

For the same problem, instead, the consistent energy equation (1.4) linearized with respect to small perturbations δh in height and δU in velocity gives

$$\frac{3}{5}\delta h_t + \frac{6}{5}\delta U_t + \frac{27}{35}\delta h_x + \frac{81}{35}\delta U_x + Fr^{-2}\delta h_x - \frac{3}{Re}(\delta h + \delta U) = -\frac{3}{Re}(2\delta U - \delta h). \quad (3.1)$$

For sinusoidal disturbances proportional to $\mathrm{e}^{\mathrm{i}k(x-ct)}$, (3.1) together with the linearized continuity equation $\delta h_t + \delta h_x + \delta U_x = 0$ produces the dispersion relation

$$\frac{6}{5}c^2 - \frac{102}{35}c + \frac{54}{35} - \frac{1}{Fr^2} = \frac{3}{\mathrm{i}kRe}(c-3). \quad (3.2)$$

When kRe is small (as it must be), one of the two roots of (3.2) can be expanded as $c \simeq 3 + (\mathrm{i}kRe/3)(Fr_c^{-2} - Fr^{-2})$ with the critical Froude number $Fr_c = \sqrt{5/18} \simeq 0.53$, which are precisely (care taken of each paper's notation) Benjamin's equations (5.3) and (5.5) or Yih's equations (37) and (38). The other root of the dispersion relation corresponds to a strongly damped mode.

In comparison, the inconsistent momentum equation (1.3) gives $Fr_c = 1/\sqrt{3} \simeq 0.58$ while the plug-flow one-dimensional formulation gives $Fr_c = 0.5$, both not far away but not right either.

4. The general three-dimensional case

It should be clear that the use of the one-dimensional energy equation as a consistent first-order approximation is not restrained to two-dimensional flow but applies equally well to a circular pipe, or for that matter to any cross-section shape. This includes dilatable pipes, for instance in biological applications (Grotberg & Jensen 2004), where wall position and pressure become coupled through wall elasticity, as well as free-surface flows such as the falling film. The only assumption necessary for validity of a minimum-dissipation property analogous to (2.3) is that either velocity or shear stress be zero on the lateral boundary, which can thus consist of any mix of solid and free surfaces. (For dilatable pipes it suffices that any possible longitudinal wall velocity be $O(\varepsilon^2)$, as is often the case when it is elastically generated.)

With reference to a general cross-section of area $S(x, t)$, the averaged kinetic-energy equation,

$$\frac{\partial}{\partial t}\frac{\alpha S U^2}{2} + \frac{\partial}{\partial x}\frac{\beta S U^3}{2} + SUP_x = -\frac{\gamma U^2}{Re}, \quad (4.1)$$

with the zeroth-order correction coefficients

$$\alpha = \frac{1}{SU^2}\int u^{(0)2}\,\mathrm{d}S, \quad \beta = \frac{1}{SU^3}\int u^{(0)3}\,\mathrm{d}S, \quad \gamma = \frac{1}{U^2}\int \left(\nabla u^{(0)}\right)^2\,\mathrm{d}S, \quad (4.2)$$

is consistently valid up to first order in a slowness parameter ε for all two- or three-dimensional closed-duct or open-channel flows. The section-averaged momentum equation, on the other hand, is inconsistent with a zeroth-order expression of shear stress. However, it can be put to good use if written in the form

$$\frac{\partial}{\partial t}(SU) + \frac{\partial}{\partial x}(\alpha SU^2) + SP_x = -\oint (\tau^{(0)} + \varepsilon\tau^{(1)})\,\mathrm{d}c. \tag{4.3}$$

After P_x is extracted from (4.1), (4.3) provides the $(0+1)$th-order sidewall-integrated shear stress, which could only have been obtained otherwise from a complete three-dimensional solution for $u^{(1)}$.

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